SYMPLECTIC GEOMETRY PRECIS

In preparation for the Qualifying Examination.

Symplectic Manifolds

Basic concepts:

(M, ω) symplectic iff $\dim(M) = 2n$, ω is a closed, nondegenerate 2-form

$\hookrightarrow$ $\omega_p : T_pM \times T_pM \to \mathbb{R}$ symplectic

Thm: ω symplectic $\Leftrightarrow$ $\omega^n \neq 0$.

Maslov Index

Symplectic matrix $A = \mathcal{U}P$, $\underset{\text{unitary}}{\mathcal{U}} = A(A^TA)^{-1/2}$, $P = (A^TA)^{1/2}$

Hom equiv. $\varphi_t : Sp(2n, \mathbb{R}) \to \mathcal{U}(n)$; $\varphi_t(A) = A(A^TA)^{-t/2}$

Can show that $\pi_1(\mathcal{U}(n)) = \mathbb{Z}$ by $\det_{\mathbb{C}} : \mathcal{U}(n) \to S^1$. Maslov index is an explicit description of this.

$\varphi : \mathbb{R}/\mathbb{Z} \to Sp(2n)$

homotopy, product, direct sum and normalization

flow through $\mathcal{U}(n)$ and take degree.

Lagrangian Subspaces

$\mathcal{L}(n) = \mathcal{L}(\mathbb{R}^{2n}, \omega_0)$

$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$, $\Lambda = \mathrm{im}(Z)$. $\Lambda \in \mathcal{L}(n) \Leftrightarrow X^TY = Y^TX$. $+ \ \mathrm{rk}(Z) = n$

Lagrangian frame unitary: $\tilde{Z} := X + iY$ unitary.

Prop: • $\Lambda \in \mathcal{L}(n)$, and $A \in Sp(2n)$, $A\Lambda \in \mathcal{L}(n)$

• $\Lambda, \Lambda' \in \mathcal{L}(n)$, $\exists$ $A \in \mathcal{U}(n)$ s.t $\Lambda = A\Lambda'$.

Can build a symplectically standard and orthogonal basis using Lagrangians.

Examples:

1) $(\mathbb{R}^{2n}, \omega_0)$, $\omega_0 = \sum_j dx_j \wedge dy_j$

2) $S^2 = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$

$\omega_x(\xi, \eta) = \langle x, \xi \times \eta \rangle$

Symplectic Vector Bundle

(E, ω), $\pi : E \to M$, $\omega \in \Gamma(E)$ a "symplectic" form. Equivalent to the reduction of the structure group to $Sp(2K)$.

Example: $E \to M$ any v.b, then $E \oplus E^*$

$\Omega_{can}(v_0 \oplus v_1^*, w_0 \oplus w_1^*) = w_1^*(v_0) - v_1^*(w_0)$.

Moser and Darboux:

Useful lemma: $\frac{d}{dt} \rho_t^* \omega_t = \rho_t^* \left(\frac{d}{dt}\omega_t + \mathcal{L}_{v_t}\omega_t \right)$

Tubular Neighbourhood Theorem:

$i: X \hookrightarrow M$, then $\exists$ convex nhood U_0 of X in NX, and nhood U of $X \subseteq M$, and a diffeomorphism $\varphi: U_0 \to U$:

$$\begin{tikzcd} NX \supseteq U_0 \arrow[rr, "\varphi"] & & U \subseteq M \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

Homotopy Invariance

let $\rho_t: M \circlearrowleft$, ω closed deg k, then $\rho_t^*\omega$ also closed.
How are $[\rho_t^*\omega]$ and $[\omega]$ related?

↳ $\rho_1^* - id = dQ - Qd \Rightarrow [\rho_1^*\omega] = [\omega]$

Difference is exact.

Moser's Theorem

M compact, $[\omega_0] = [\omega_1]$, and $\omega_t = (1-t)\omega_0 + t\omega_1$ symplectic $\forall\ t \in [0,1]$, then $\exists!\ \rho: M \times \mathbb{R} \to M$, s.t $\rho_t^*\omega_t = \omega_0$ and $\varphi_0 = id$.

Sketch of proof:

Prove it for $M = \mathbb{R}^n$, X compact:

$\mathcal{U}^\varepsilon = \{ p \in \mathbb{R}^n \mid |p-q| < \varepsilon, \text{ some } q \in X \}$

ε small $\to \forall p\ \exists!\ q \in X$ s.t $|p-q|$ is minimal

let $\pi: \mathcal{U}^\varepsilon \to X$, $p \mapsto q$

↳ smooth submersion s.t $(1-t)p + tq \in \mathcal{U}^\varepsilon$

On the other hand,

$N_x X \cong \{ v \in \mathbb{R}^n \mid v \perp w\ \ \forall\ w \in T_x X \}$.

let $exp: NX \to \mathbb{R}^n$, $(x,v) \mapsto x+v$ $\qquad NX^\varepsilon = \{(x,v) \in N_x X \mid |v| < \varepsilon\}$

ε small:

$$\begin{tikzcd} NX^\varepsilon \arrow[rr, "exp"] & & \mathcal{U}^\varepsilon \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

For compact X mani, replace $|.|$ with Riemman norm.

Moser Trick:

On a compact manifold, equivalent to finding a family of v.f.

If such a ρ_t exists, then $0 = \frac{d}{dt}(\rho_t^*\omega_t)$

$= \rho_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right)$

$\Rightarrow \mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0$

Homotopy formula $\Rightarrow \frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\mu$

$\dots \Rightarrow \iota_{X_t}\omega_t + \mu = 0$

Moser equation: $\iota_{X_t}\omega_t + \mu = 0$

Relative Moser

$i: X \hookrightarrow M$, X compact, ω_0, ω_1 symplectic on M s.t $\omega_0|_p = \omega_1|_p$ $\forall p \in X$. Then $\exists\ \mathcal{U}_0, \mathcal{U}_1 \subseteq M$, $\varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i_0"] \arrow[ur, "i_1"'] & \end{tikzcd} \qquad \varphi^* \omega_1 = \omega_0$$

Application: Darboux

(M, ω), let $p \in M$. Then $\exists\ (\mathcal{U}, x^i, y^i)$ on p s.t on $\mathcal{U}$,

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

Weinstein Lagrangian Neighbourhood Theorem

Statement: $i: X^n \hookrightarrow M^{2n}$, ω_0 and ω_1 symplectic, and $i^*\omega_0 = i^*\omega_1 = 0$. Then $\exists\ \mathcal{U}_0, \mathcal{U}_1$ of X, $\varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd} \qquad \varphi^* \omega_1 = \omega_0$$

Proof: uses Relative Poincaré Lemma:

$X \subseteq \mathcal{U} \subseteq M$, $i: X \to M$, ω closed on $\mathcal{U}$, $i^*\omega = 0$.

$\Rightarrow \omega$ exact and $\omega = d\mu$, $\mu \in \Omega^{k-1}(\mathcal{U})$, $\mu_x = 0$ $\forall x \in X$

Proof sketch

Choose nhood $\mathcal{U}$ of X, $\omega_1 - \omega_0$ closed and vanishes on X, $\Rightarrow$ Relative poincaré $\Rightarrow \exists\ \mu \in \Omega^1(\mathcal{U})$ s.t $\omega_1 - \omega_0 = d\mu$, $\mu_p = 0$ $\forall p \in X$.

Let $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t\,d\mu$ $\}$ symp on small $\mathcal{U}$

Then apply Moser

Proof sketch:

Apply relative Moser to $X = \{p\}$:

1) On T_pM, $\omega_p = \sum_i dx^i|_p \wedge dy^i|_p$

$\hookrightarrow$ extend $\tilde{\omega} = \sum_i dx^i \wedge dy^i$

2) $\tilde{\omega}$ and ω agree on X, $\exists\ \varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t

$\varphi^* \tilde{\omega} = \omega \Rightarrow \sum d(x^i \circ \varphi) \wedge d(y^i \circ \varphi) = \omega$

3) set $\bar{x}^i = x^i \circ \varphi$, $\bar{y}^i = y^i \circ \varphi$.

Two Important Prelim Results:

Technical: have $i^*\omega_1 = i^*\omega_0 = 0$ on X, but does not say $\omega_1|_p = \omega_0|_p$ $\forall p \in X$. Work around: perturb ω_1 a little so that it agrees $\forall p \in X$, then apply Moser.

Prop: (V^{2n}, Ω_0), (V^{2n}, Ω_1) symplectic vector space, $\mathcal{U}$ Lagrangian for Ω_0, Ω_1, W any complement to $\mathcal{U}$ in V. From W we can canonically construct a linear iso $L: V \xrightarrow{\sim} V$ s.t $L|_{\mathcal{U}} = id|_{\mathcal{U}}$, $L^*\Omega_1 = \Omega_0$

Whitney Extension theorem: suppose $\forall p \in X$ we have a linear iso $L_p: T_pM \xrightarrow{\sim} T_pM$ s.t $L_p|_{T_pX} = id$ and L_p depends smoothly on p. Then $\exists$ embedding $h: N \hookrightarrow M$, $X \subseteq N$ s.t $h|_X = id_X$, $dh_p = L_p$ $\forall p \in X$.

WLNT PROOF:

Whitney $\Rightarrow \exists$ nhood N of X, $h: N \hookrightarrow M$ s.t $h|_X = id_X$, $dh_p = L_p$ $\forall p \in X$. Then $\forall p \in X$,

$(h^*\omega_1)_p = (dh_p)^*\omega_1|_p = L_p^*\omega_1|_p = \omega_0|_p$

Apply Moser to ω_0 and $h^*\omega_1$ on $i: X \hookrightarrow N$. $\to f: \mathcal{U}_0 \to N$

Let $\varphi = h \circ f$.

Contact Structures

Contact manifold (M,ξ) : $\dim(M) = 2n+1$, ξ a codim 1 dis^{n} that is maximally non-integrable. Say $\xi = \ker(\alpha)$.

Frobenius says $d\alpha \equiv 0$ on integrable

↳ contact is when $d\alpha$ is nondegenerate.

↳ $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) + \alpha([X,Y])$

Properties of contact forms

1) $d\alpha$ nondeg $\Leftrightarrow$ $(d\alpha)^n$ vol form on ξ $\Leftrightarrow$ $\alpha \wedge (d\alpha)^n \neq 0$.

2) $\xi = \ker(\alpha) = \ker(\alpha')$ $\Rightarrow$ α contact $\Leftrightarrow$ α' contact

↳ $\alpha = f\alpha'$, $f > 0$

3) $d\alpha$ independent of α up to +ve scaling

Legendrian: $L \subset (M,\xi)$ Legendrian if $T_qL \subseteq (\xi_q, d\alpha_q)$ Lagrangian subspace

Integral submanifolds $L \subseteq M$ s.t $T_qL \subseteq \xi_q$ are isotropic.

Reeb vector field: $T_qM = \ell_q \oplus \xi_q$

$\xi_q = \ker(\alpha_q)$, $\ell_x = \ker(d\alpha_q)$

$\exists!$ v.f $R_\alpha : M \to TM$ s.t. $\iota(Y)d\alpha = 0$, $\alpha(Y) = 1$

Example:

$M = \mathbb{R}^3$, $\alpha = u \cdot dx$ is contact $\Leftrightarrow$ $\rho = (\nabla \times u)\cdot u$ is never 0

$d\alpha(v_1, v_2) = ((\nabla \times u) \times v_1) \cdot v_2$

Contact vector field associated with H:

$\rho X_H = u \times \nabla H + H(\nabla \times u)$.

Contact Manifold Examples:

1) $\mathbb{R}^{2n+1}$, $\alpha_0 = \sum_j y_j\, dx_j - dz$

$d\alpha = \sum_j dy_j \wedge dx_j$

$\xi_q = \text{span}\left\{ \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial z_j}, \frac{\partial}{\partial y_j} \right\}$

$R_\alpha = -\frac{\partial}{\partial z}$

Legendrian: planes parallel to y axis.

2) $\mathbb{R}^{2n+1}$, $\alpha_1 = \frac{1}{2}\sum_j (y_j\, dx_j - x_j\, dy_j) - z$

isomorphic to 1)

3) L any compact manifold: 1-jet bundle:

$J^1 L := T^*L \times \mathbb{R}$

is a contact manifold w/ contact form

$\alpha = \lambda_{can} - dz$

$R_\alpha = -\frac{\partial}{\partial z}$

Legendrian: $L_S = \{ (x, dS(x), S(x)) \mid x \in L \} \subset J^1L$.

for any $S \in C^\infty(L)$.

4) Riemannian L, $M = S(T^*L)$ be the unit sphere bundle in T^*L.

$\alpha = \lambda_{can}|_{S(T^*L)}$

is a contact form.

R_α = Hamiltonian v.f dual to geodesic flow on TL.

5) $\Sigma = S^{2n+1} \subseteq \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. $\forall q \in \Sigma$,

$\xi_q := T_q\Sigma \cap J_0 T_q \Sigma$

has codim 1

$\lambda_0 = \frac{1}{2}\sum_{j=0}^{n} y_j\, dx_j - x_j\, dy_j$

Reeb flow: $\dot{x} = 2y$, $\dot{y} = -2x$.

Contactomorphisms:

Contactomorphism: (M,ξ), α contact form. $\varphi : M \to M$ a diffeo s.t. $\varphi^*\alpha = e^h \alpha$, $h \in C^\infty(M)$.

Contact isotopy: $\varphi_t : M \to M$, $\varphi_0 = id$, $\varphi_t^*\alpha = e^{h_t}\alpha$.

Contact vector fields:

(1) $X : M \to TM$ is contact iff $\exists\ H : M \to \mathbb{R}$ s.t

$$\iota_X \alpha = H, \qquad \iota_X d\alpha = (\iota_R dH)\alpha - dH \qquad (\dagger)$$

(2) This is invertible: any $H \in C^\infty(M)$ gives ! v.f.

Important Exercise **Remarks:**

1) Not every contact v.f is the Reeb v.f of a contact form

X is Reeb v.f of $\alpha' \iff X$ is transverse to ξ

i.e. $\alpha(X) \neq 0\ \forall$ defining α

2) β a 1-form s.t $\beta(R) = 0$, then $\exists!$ v.f $\subseteq \xi$ such that $\beta = \iota_X d\alpha$.

Contact Isotopy:

Idea: want $\alpha_t \rightsquigarrow \varphi_t$ s.t $\varphi_t^*\alpha_t = f_t \alpha_0 \quad (*)$, $\varphi_0 = id$.

look for vf: $\frac{d}{dt}\varphi_t = X_t \circ \varphi_t$, $\varphi_0 = id$.

$$\frac{d}{dt}(*) = \varphi_t^*\left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t\right) = g_t \varphi_t^*\alpha_t \qquad g_t := \frac{1}{f_t}\frac{d}{dt}f_t$$

find X_t, $h_t = g_t \circ \varphi_t^{-1}$ s.t

$$\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t = h_t \alpha_t$$

To solve: let $h_t = \iota_{R_t}\frac{d}{dt}\alpha$. Then $h_t\alpha_t - \frac{d}{dt}\alpha_t$ vanishes on $\mathcal{Y}_t$. Then $\exists!\ X_t$ s.t

$$\iota_{X_t} d\alpha_t = h_t\alpha_t - \frac{d}{dt}\alpha_t \qquad \alpha_t(X_t) = 0.$$

Proof of Contact v.f. Equivalence:

$(\dagger)$ holds $\Rightarrow \mathcal{L}_X\alpha = g\alpha$, $g = \iota_R dH \Rightarrow X$ contact

X contact $\Rightarrow \mathcal{L}_X\alpha = g\alpha$, let $H = \iota_X\alpha$

$\Rightarrow \iota_X d\alpha = g\alpha - H \to$ eval on R to get $g = \iota_R dH$.

(ii) let $H \in C^\infty(M)$. $\exists!$ section $Z \subseteq \xi$ s.t $-\iota_Z d\alpha|_\xi = dH|_\xi$.

let $X_H = HY + Z$.

Contact Darboux:

Thm: Every contact structure is locally diffeomorphic to the standard structure.

proof:

1) pick a point $p \in M$. Look at $T_pM = \xi_p \oplus \ell_p$. Then for a pair α_p, $d\alpha_p$, one can choose a basis so that $\langle \partial_{x^i}, \partial_{y^i}\rangle = \xi_p$, $\langle \partial_z \rangle = \ell_p$, and $\partial_{x^i}, \partial_{y_i}$ are symplectically standard.

2) Then these give a chart $\phi : U \to \mathbb{R}^{2n+1}$, $\{x^i, y^i, z\}$ for which $\phi^*\alpha_0|_p = \alpha|_p$, but not necessarily everywhere else.

3) Interpolate $\alpha_t = t\phi^*\alpha_0 + (1-t)\alpha$, all contact

4) Gray's $\to \varphi_t^*\alpha_t = f_t\alpha_0$, so $\phi\circ\varphi : U \to \mathbb{R}^{2n+1}$ is the appropriate chart.

Symplectization

(M, ξ) contact. Then

$$W = \left\{ (q, v^*) \;\middle|\; \begin{array}{l} q \in M,\ v^* \in T_q M,\ \ker v^* = \xi_q, \\ v \in T_q M,\ \alpha_q(v) > 0 \Rightarrow v^*(v) > 0 \end{array} \right\}$$

$W \subseteq (T^*M, \omega_{can})$:

$W_\alpha = \mathbb{R} \times M$, $\omega_\alpha = -d(e^s \alpha)$

Then $\iota_\alpha : \mathbb{R} \times M \to W$; $(s, q) \mapsto e^s \alpha_q$

is a symplectomorphism $(W_\alpha, \omega_\alpha) \to (W, \omega_{can}|_W)$.

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Gray's Stability Theorem:

α_t contact on a closed manifold $\Rightarrow$ $\alpha_t = \varphi_t^*(f_t \alpha_0)$.

1) Equivalent to show $\varphi_t^* \alpha_t = f_t \alpha_0$.
2) Let $h_t = \iota_{R_t} \frac{d}{dt} \alpha_t$, and let $\sigma_t = \frac{d}{dt}\alpha_t - h_t \alpha_t$

   Then $\sigma_t \in \ker(R) \Rightarrow \sigma_t$ is tangent to ξ.
3) Invert σ_t to get $X_t \subseteq \xi_t$ $\rightarrow$ $\iota_{X_t} d\alpha_t = \sigma_t$, $\iota_{X_t}\alpha_t = 0$.
4) Satisfies $$\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t = h_t \alpha_t$$
5) Flow of $X_t = \varphi_t$ $\leftarrow$ M compact + smooth.
   $$\frac{d}{dt}\varphi_t = X_t \circ \varphi_t$$
6) Set $f_t = e^{\int_0^t h_t \circ \varphi \, dt}$
7) Check $\frac{d}{dt}\left(f_t \varphi_t^* \alpha_t\right) = 0$.

Examples:

1) Unit Sphere $M = S^{2n-1}$, $\alpha = \frac{1}{2}\sum_{j=1}^{n} y_j dx_j - x_j dy_j$

   symplectization $\simeq \mathbb{C}^n \setminus \{0\}$ with standard form

   $$\mathbb{R} \times S^{2n-1} \to \mathbb{C}^n \setminus \{0\} : (s, x, y) \mapsto e^{s/2}(x + iy).$$

2) Euclidean Space $M = \mathbb{R}^{2n+1}$, $\alpha = \sum_{j=1}^{n} y_j dx_j - dz$

   Symplectization $\cong (\mathbb{R}^{2n+2}, \omega_0)$

   $\mathbb{R} \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+2} : (s, x, y, z) \mapsto (s, e^s z, x_1, e^s y_1, \dots, x_n, e^s y_n)$

3) Torus unit cotangent bundle

   $\hookrightarrow$ parallelizable b.c Lie group. $M = \mathbb{T}^n \times S^{n-1}$

   $\alpha = \sum_{j=1}^{n} y_j dx_j$

   Symplectization $\cong \mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\})$, ω_{can}

   $\mathbb{R} \times \mathbb{T}^n \times S^{n-1} \to \mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\}) : (s, x, y) \mapsto (x, e^s y)$.

Correspondence contact $\leftrightarrow$ symplectic

$(W_\alpha, \omega_\alpha)$ Extrinsic symplectization

1) $f > 0$, $f\alpha$ also contact
$(W_\alpha, \omega_\alpha) \simeq (W_{f\alpha}, \omega_{f\alpha})$: $(s,q) \mapsto (s - \log(f(q)), q)$

2) $L \subset M$ legendrian $\Leftrightarrow$ $\mathbb{R} \times L$ Lagrangian in W_α

3) $\varphi: M \to M$ contactomorphism, $\varphi^* \alpha = e^h \alpha$
$\Leftrightarrow$ $\tilde{\varphi}(s,q) = (s - h(q), \varphi(q))$ a symplectomorphism

Symplectic $\to$ Contact.

Liouville vector field: (W, ω) X s.t

$$\mathcal{L}_X \omega = d\iota(X)\omega = \omega$$

flow: $\varphi_t^* \omega = e^t \omega$ wherever defined.

Idea: $M^{2n-1} \subseteq (W^{2n}, \omega)$ $\to$ when $\varphi: (W_\alpha, \omega_\alpha) \xrightarrow{\text{symp?}} (U, \omega)$

Idea: need a vector field X that acts like $\frac{\partial}{\partial s}$ and carries symplectic structure:

$$(\phi_t^{\partial s})^* e^s \alpha = e^{s+t} \alpha$$

$$\Rightarrow (\phi_t^{\partial s})^* \omega_\alpha = e^t \omega_\alpha$$

want $(\phi_t^X)^* \omega = e^t \omega$

Then $\alpha = \iota_X \omega$ satisfies $d\alpha' = \omega$

$\Rightarrow$ $\alpha(X) = 0$ $\Rightarrow$ $\mathcal{L}_X \alpha = \alpha$, $(\phi_t^X)^* \alpha = e^t \alpha$

Prop: (W, ω) symplectic, $M \subset W$ contact hypersurface. TFAE:

(1) $\exists$ contact form α on M s.t $-d\alpha = \omega|_M$

(2) $\exists$ Liouville v.f $X: U \to TW$, $M \subset U \subset W$, s.t $X \pitchfork M$

Pf: (2) $\Rightarrow$ (1): $\alpha := -\iota(X)\omega$

Almost Complex Structures

$J: TM \to TM$, $J^2 = -\mathrm{id}$

$\Rightarrow$ M always oriented w/ even dim

(M, ω) symplectic: **ω-tame:** $\omega(v, Jv) > 0 \ \forall J$

ω-compat: $\omega(Jv, Jw) = \omega(v, w)$.

↳ $\langle v, w \rangle := \omega(v, Jw)$ = inner product.

Almost compatible structures + Levi-Civita:

$(\nabla_v J) J + J(\nabla_v J) = 0$

$\langle (\nabla_u J) v, w \rangle + \langle v, (\nabla_u J) w \rangle = 0$

$d\omega(u, v, w) = \mathfrak{S} \langle (\nabla_u J) v, w \rangle$

ω closed: $(\nabla_{Jv} J) = -J(\nabla_{Jv} J)$

Integrability:

holo charts for which J looks like J_0 (complex mult.) in local coords

$\phi: U \to \phi(U) \subset \mathbb{R}^{2n}$, $d\phi_q \circ J_q = J_0 \circ d\phi_q$

Nijenhuis Tensor:

$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$

Integrability Theorem: J integrable $\Leftrightarrow N_J = 0$.

Rem: a.c.s $\to$ maybe no local a.c. diffeos.

ϕ_t preserves $J \Leftrightarrow \mathcal{L}_X J = 0$

$\Leftrightarrow \mathcal{L}_X \circ J = J \circ \mathcal{L}_X$,

$\Leftrightarrow [X, JY] = J[X, Y]$ ✓ may not be an X with this property.

Theorem: every a.c.s on a 2-dim manifold is integrable

Kahler Manifolds

(M, J, ω), J is ω-compatible

Example: in dim $n = 1, 3, 7$, every oriented hypersurface inherits an almost complex structure

$J_x u := \nu(x) \times u$

$\omega_x(u, v) = \langle J_x u, v \rangle = \langle \nu(x), u \times v \rangle$

Ex 1: $(\mathbb{R}^{2n}, J_0, \omega_0)$ $\{x_1, \dots, x_n, y_1, \dots, y_n\}$.

$z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$

$\omega_0 = \frac{i}{2} \partial \bar{\partial} f = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ $\quad f(z) = \sum_{j=1}^{n} z_j \bar{z}_j$

Ex 2: $\mathbb{CP}^n = \{[z_0 : \dots : z_n]\}$, trans. are holo $\Rightarrow$ mult by i is a valid integrable c.s. Fubini-Study form is compatible with J.

Circle Actions

Hamiltonian Circle Action: 1-parameter family $\mathbb{R} \to \mathrm{Symp}(M) : t \mapsto \varphi_t$, 1-periodic, $\varphi_0 = \varphi_1 = id$. integral of Hamiltonian v.f. X_H.

$H =$ "moment map"

Lem: $S^1 \overset{free}{\curvearrowright} H^{-1}(\lambda)$, λ regular. Then

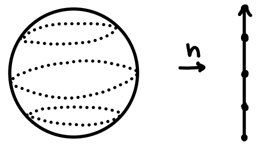
$$B_\lambda := H^{-1}(\lambda)/S^1$$

has a symplectic form τ_λ s.t. $q^* \tau_\lambda = \omega|_{H^{-1}(\lambda)}$.

called the symplectic quotient of (M, ω) at λ.

Condition for S^1 actions to be Hamiltonian

Example:



$h : S^3 \to \mathbb{R}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto 2\pi z$

Example: $\mathbb{CP}^n$ and Fubini-Study Form:

$S^1 \curvearrowright (\mathbb{R}^{2n+2}, \omega_0)$, by $\times e^{2\pi i t}$ on $\mathbb{C}^{n+1}$. This is gen. by the function $H(z) = -\pi |z|^2$.

Then the symplectic quotient at $\lambda = -\pi$ is $\mathbb{CP}^n$ w/ Fubini-Study form.

$\pi_H : \mathbb{C}^{n+1} \to \mathbb{CP}^n \rightsquigarrow \pi_H : S^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration.

$\mathbb{CP}^1$: on $\mathcal{U}_0 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_0 \neq 0\}$ is given by

$$\omega_{FS} = \frac{dx \wedge dy}{(x^2+y^2+1)^2}, \quad \frac{z_1}{z_0} = z = x + iy.$$

J-holo curves

Riemann surface $(\Sigma, j_\Sigma, dvol_\Sigma)$

J-holo curve: $u : \Sigma \to M$ s.t du is complex linear wrt j and J: $J \circ du = du \circ j$.

$\Leftrightarrow \bar{\partial}_J(u) = 0$, $\bar{\partial}_J(u) = \frac{1}{2}(du + J \circ du \circ j) \in \Omega^{0,1}(\Sigma, u^* TM)$

Conformal coords $z = s + it$ on Σ,

$$\bar{\partial}_J(u) = \frac{1}{2}(u_s + J u_t)\, ds + \frac{1}{2}(u_t - J u_s)\, dt$$

$\Rightarrow$ u j-holo $\Leftrightarrow$ $u_s + J u_t = 0$

Critical points:

lem: $u : \Sigma \to M$, Σ closed, J C^1, u nonconstant. Then

$$X = u^{-1}(\{ u(z) \mid z \in \Sigma,\ du(z) = 0 \})$$

is finite. Moreover, $u^{-1}(x)$ is finite $\forall x$.

pf: idea is that ∞-jet at u is nonzero, so $\exists$ $\ell \in \mathbb{N}$ s.t $u(z) = O(|z|^{\ell})$, and $u(z) \neq O(|z|^{\ell+1})$

$\Rightarrow J(u(z)) = J_0 + O(|z|^{\ell})$, and $u(z) = a z^{\ell} + O(|z|^{\ell+1})$.

$$\partial_s u + J(u)\, \partial_t u = 0 \Rightarrow \partial_s T_\ell(u) + J_0\, \partial_t T_\ell(u) = 0.$$

$u(z) = a z^{\ell} + O(|z|^{\ell})$, $\partial_s u(z) = \ell a z^{\ell-1} + O(|z|^{\ell})$

Then $0 < |z| < \varepsilon$, $u(z) \neq 0$, $du(z) \neq 0$.

Important results:

1) if two J-holo points sequences converging to the same regular point, then in a neighbourhood of zero they must be related by a holo fct.

2) J C^2, Σ_0, Σ_1 closed. $u_i : \Sigma_i \to M$ J-holo, with $u_0(\Sigma_0) \neq u_1(\Sigma_0)$ nonconstant. Then $u_0^{-1}(u_1(\Sigma_1))$ is at most countable and can only accumulate at critical pts.

Example:

$(V, J) = (S_1 \times S_2, J_1 \oplus J_2)$, product of two Riemann surfaces. Then the graphs of holo maps $(S_1, J_1) \to (S_2, J_2)$ are regular J-curves.

Somewhere Injective curves.

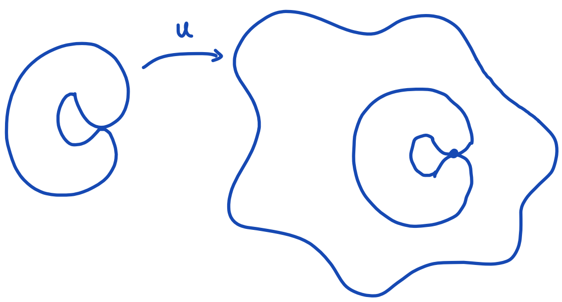
Multiply covered: (Σ, j) compact, $\exists\, (\Sigma', j')$ comp. and $u' : \Sigma' \to M$, and holo. branched covering $\phi : \Sigma \to \Sigma'$, s.t $u = u' \circ \phi$, $\deg(\phi) > 1$.

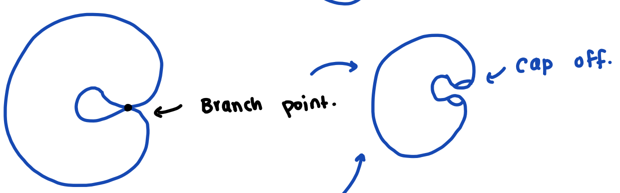
Somewhere injective: $\exists\, z \in \Sigma$ s.t $du(z) \neq 0$, $u^{-1}(u(z)) = \{z\}$.

↳ simple J-holos have injective points open and dense.

simple J-holo ⇒ somewhere injective.

idea: construct a branched cover of u that only has degree 1 ⟺ u is somewhere injective. Since u is assumed to be simple, ⇒ somewhere inj.





then $\Sigma' =$ + caps.

get $u' : \Sigma' \to M$, u' somewhere inject.
b.c. took away obstacle points.

Moduli Spaces and Transversality

$\mathcal{M}(A,\Sigma;J) = \{u \in C^\infty(\Sigma, M) \mid J \circ du = du \circ j_\Sigma,\ [u] = A\}.$

$\mathcal{M}^*(A,\Sigma;J) = \mathcal{M}(A,\Sigma;J) \cap \{\text{simple } u\}$

I) Description of $\mathcal{M}$ using ∞-dim bundles.

$\mathcal{B} = \{u : \Sigma \to M,\ [u] = A\},$

$T_u\mathcal{B} = \Omega^0(\Sigma, u^*TM)$ ← smooth v.f $\in TM$ along u.

$\mathcal{E} \to \mathcal{B}$

$\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$ ← vector valued antilinear forms

Section $S(u) := (u, \bar\partial_J(u))$

$\mathcal{M}(A;\Sigma;J) = S \cap S_0.$

II) The operator D_u

D_u = vertical differential of S at u.

$D_u = \pi_u \circ dS$

$T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u$ $\quad dS : T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$

Transverse $\cap \Leftrightarrow D_u$ surjective

In local (conformal coordinates)

$D_u\xi = \bar\partial_J\xi - \underbrace{\tfrac{1}{2}(J\partial_\xi J)(u)\,\partial_J(u)}_{\text{also antiholo 1-form}}$

↓

real Cauchy-Riemann operator.

III General description of operator

need to split $T_{(u,\bar\partial_J(u))}\mathcal{E}$ into horizontal + vertical dist^{n}s.

∇ Levi-Civita on TM

$\tilde\nabla_v X = \nabla_v X - \tfrac{1}{2}J(\nabla_v J)X$

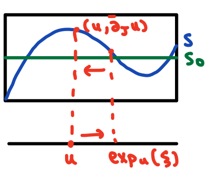
$\xi \in \Omega^0(\Sigma, u^*TM)$

$\Phi_u(\xi) : u^*TM \to \exp_u(\xi)^*TM$

parallel transport along geodesics $s \mapsto \exp_{u(z)}(s\xi(z))$.

set $\mathcal{F}_u = \Phi_u(\xi)^{-1}\,\bar\partial_J(\exp_u(\xi))$

Picture :



$D_u\xi = d\mathcal{F}_u(0)\xi.$

IV) Properties of D_u

D_u is an $\mathbb{R}$-linear, Cauchy-Riemann operator, hence is Fredholm.

Fredholm : dim(ker), dim(coker) $< \infty$.

Riemann-Roch : $\text{ind}(D_u) = n(2-2g) + 2c_1(u^*TM).$

genus of g

V) $\mathcal{M}$ is a manifold for "generic" J.

$\mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell) := \{(u,J) \mid J \in \mathcal{J}^\ell, u \in \mathcal{M}^*(A,\Sigma;J)\}$

is Banach for large enough ℓ

$\pi : \mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell) \to \mathcal{J}^\ell$ is Fredholm

$\Rightarrow$ Sard-Smale $\Rightarrow$ regular values of π is of 2^{nd} category.

Use $\ell < \infty$ to have Banach manifold, Fredholm + Sard-Smale.

Sobolev Spaces

Idea of Proof of Manifold Structure

$\mathcal{F}_u : W^{k,p}(\Sigma, u^*TM) \to W^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes u^*TM).$

Basically what $\mathcal{F}_u$ was before.

$\mathcal{F}_u^{-1}(0) = \{$ v.f s.t parallel transporting along them gives $0\}.$

↳ then $\xi \mapsto \exp_u(\xi)$ gives diffeo

$\mathcal{F}_u^{-1}(0) \simeq$ open n-hood of u in $\mathcal{M}^*(A,\Sigma;J).$

Let J be regular : $d\mathcal{F}_u(0) = D_u$ is surjective

and $\mathcal{F}_u$ Fredholm between Banach spaces, then

∞-dim Implicit Function Theorem $\Rightarrow \exists$ neighbourhood

of 0 in $W^{k,p}(\Sigma, u^*TM)$ $\mathcal{U}$ s.t $\mathcal{U} \cap \mathcal{F}^{-1}(0)$ is a subman.

of dim $= \text{ind}(\mathcal{F}_u) = n(2-2g) + c_1(A).$

Then use diffeo to get chart in $\mathcal{M}^*(\Sigma, A; J).$

That deals with the C^ℓ case → what about C^∞?

Answer: Elliptic Bootstrapping

(Elliptic regularity) J a.c.s, C^ℓ, $\ell \geq 1$. $u : \Sigma \to M$ J-holo of class $W^{1,p}$, $p > 2$, then u of class $W^{\ell+1,p}$. In particular, u is of class C^ℓ, and if J is smooth then so is u.

Proof that Regular J are generic

Key ingredient: the projection $\pi: \mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell) \to \mathcal{J}^\ell$.

tangent space of Moduli space:

$$T_{(u,J)}\mathcal{M}^*(A,\Sigma;J^\ell) \subset \underbrace{W^{1,p}(\Sigma, u^*TM)}_{T \text{ of } \mathcal{M}^*(\Sigma,A;J)} \times \underbrace{C^\ell(M, \mathrm{End}(TM,J,\omega))}_{\text{think of as tangent vectors to } J \text{ of } u.}$$

Comprises pairs (ξ, Y) s.t

$$D_u\xi + \frac{1}{2}Y(u)\circ du \circ j_\Sigma = 0$$

Note: $d\pi: T_{(u,J)}\mathcal{M}(A,\Sigma;J^\ell) \to T_J J$
$(\xi, Y) \mapsto Y$

fix (u,J) for a moment!

$\ker(d\pi) = \xi \in T_u\mathcal{M}^*(A,\Sigma;J) = \operatorname{Ker} D_u$

Deals w/ ker, coker & that of D_u by some Linear Alg.

So since D_u is Fredholm, $d\pi$ fredholm.

Then use Sard-Smale.

Compactness

I) Energy

$E(u) = \frac{1}{2}\int_\Sigma u^*\omega = \frac{1}{2}\int_\Sigma |du(z)|^2 \, d\mathrm{vol}_\Sigma$

↳ on a closed Σ → topological invariant.

uniform bound on du: $\sup_\nu \|du^\nu\|_{L^\infty(K)} < \infty$

⇒ convergence to J-holo curve.

case when $\sup_\nu \|du^\nu\|_{L^\infty(K)} = \infty$, but

$\sup_\nu E(u^\nu) < \infty$,

then get formation of bubbles.

Proof:

z^ν be point where du^ν attains its maximum,

and let $\|du^\nu\|_{L^\infty} = |du^\nu(z^\nu)| =: c^\nu$.

pass to a subsequence → $z^\nu \to z_0$, $c^\nu \to \infty$.

idea: basically look locally at a chart of Σ

$\phi : \Omega \to \Sigma$ s.t $\phi(0) = z_0$,

$$\phi^* d\mathrm{vol}_\Sigma = \lambda^2 \, ds \wedge dt,$$

$$\lambda : \Omega \to (0,\infty), \quad \lambda(0) = 1, \quad \tfrac{1}{2} \le \lambda(z) \le 2.$$

Reparametrize u^ν as $u^\nu \circ \phi : \Omega \to M$,

z^ν now is $\phi^{-1}(z^\nu)$. Then

$\lim_{\nu\to\infty} z^\nu = 0$, and $c^\nu = \sup_\Omega \frac{|du^\nu(z^\nu)|}{\lambda} \to \infty$

okay so now we have a $\Omega \subset \mathbb{C}$ s.t the centre point blows up.

Idea is to show that, reparametrizing u^ν, we can describe the limit as a J-holo sphere

$v^\nu : B_{\varepsilon c^\nu} \to M$, $v^\nu(z) := u^\nu\left(z^\nu + \frac{z}{c^\nu}\right)$

looks like a small nhood.

Then $|dv^\nu(0)| \geq \frac{1}{2}$, $\|dv^\nu\|_{L^\infty(B_{\varepsilon c^\nu})} \le 2$

$$E(v^\nu; B_{\varepsilon c^\nu}) \le E(u^\nu).$$

this satisfies conditions for convergence of subsequence

limit function $v : \mathbb{C} \to M$ is J-holo with

$|dv(0)| \geq 1/2$, $0 < E(v) \le \sup_\nu E(u^\nu)$.

Then reparametrize back: $\mathbb{C}\setminus\{0\} \to M$, $z \mapsto v(1/z)$

Gromov Nonsqueezing

$\iota: B^{2n}(r) \hookrightarrow Z^{2n}(R)$ a symplectic embedding, then $r \le R$.

Outline of Proof:

(1) $\iota: B^{2n}(r) \longrightarrow B^2(R) \times \mathbb{R}^{2n-2}$

(2) Assume $Im(f) \subseteq$ compact sub. of $Int(Z^{2n}(R))$

(3) Then $\iota: B^{2n}(r) \hookrightarrow B^2(R) \times \left(\mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2}\right)$ embedding.

(4) Collapse $\partial B^2(R)$: $\iota: B^{2n}(R) \hookrightarrow S^2 \times T^{2n-2}$

Two ways: using J-holo spheres or J-holo disks.

Disks: look at $\mathcal{M}(D^2, A; J)/G$, $A = [B^2(R) \times \{pt\}]$

Cook up a J-holo disk $u: (D, \partial D) \longmapsto (\bar{Z}, \partial Z)$.
through $\phi(0) \subset Z$.

Look at $\overline{\iota(B^{2n}(r)) \cap u(D, \partial D)}$, and take u^{-1}, call it Σ.

if D^2 is a disk in $B^{2n}(r)$, then ι is a symplectomorphism on it. So, ι^{-1} is well defined.
Hence get $\tilde{u} = \iota^{-1} \circ u$

note $\bar{\Sigma} \subset Int(D^2)$, and $u(\partial\Sigma) \subset \partial B^{2n}(r)$.

Then the pair $u, \tilde{u}$ of J-holomorphic surfaces give us a way to interpolate between $B^{2n}(r)$ and $Z^{2n}(R)$.

Monotonicity lemma: $\tilde{u}: (\Sigma, 0) \to (\mathbb{C}^n, 0)$ proper, holo,
then $\forall r > 0$,

$$\int_{\tilde{u}^{-1}(B_r)} \tilde{u}^* \omega \geq \pi r^2.$$

$$\Rightarrow \int_\Sigma \tilde{u}^* \omega \geq \pi r^2.$$

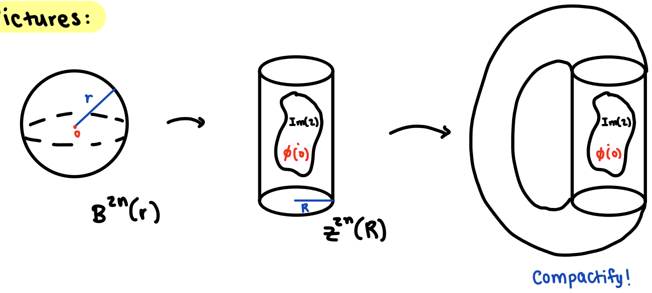
On the other hand, on Σ: $\tilde{u}^* \omega = \tilde{u}^* \iota^* \omega = (\iota \circ \tilde{u})^* \omega = u^* \omega$,

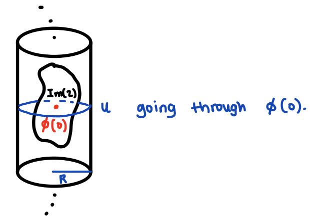
$$\Rightarrow \int_\Sigma \tilde{u}^* \omega = \int_\Sigma u^* \omega \le \int_{D^2} u^* \omega$$

$$= area(D^2)$$
$$= \pi R^2$$

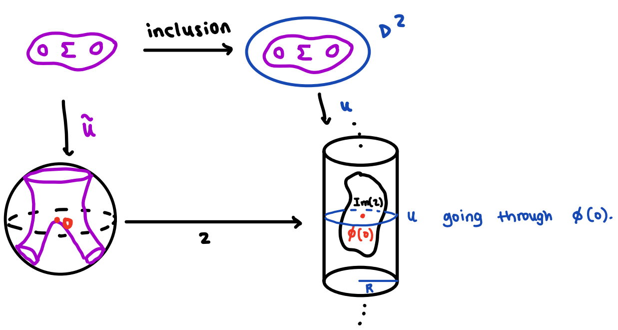
$\Rightarrow \pi r^2 \le \pi R^2 \Rightarrow r \le R.$

Pictures:





Picture:



Note: How to think of J-holo disk: